Summer Assignments
AP Calculus AB

Attached is an article from the NY Times about a mathematician and some of the work he does!

Part 1: The three explorations are all due Tuesday September 6th. Each should take about one hour to complete.

Part 2: The driving project is due the first day of school: Tuesday September 6th. This project should take about two hours. Don't come to class without it!

Instructions

- Present your work on separate paper. Be neat and organized! If necessary, rewrite your work, just as you would an essay for another class.
- Be expressive. Write in complete sentences and show your math clearly. Any reader should be able to follow your thinking and its underlying logic.
- When drawing graphs, use an appropriate window, label what the axes represent, and identify key values and points.
- At the end of each exploration, summarize your findings and conclusions in a well-written paragraph. Answer all the questions posed in the assignment and include any other insights you've made.

Convince your reader that you thoroughly understand the concepts covered!

Extra Help

Feel free to work with other students on these assignments, however YOU are responsible for learning the material not just producing answers to hand in. It often helps to share ideas. Also, I will be available via e-mail (KLaffey@rbrhs.org) most of the summer (with the exception of the first week of July.)

The New York Times
The Wild Side
Olivia Judson

May 19, 2009, 8:26 pm

Guest Column: Math and the City

Thanks again to Leon Kreitzman for four fascinating articles about biological clocks in everything from peonies to people. My sabbatical is rapidly drawing to a close — but it isn't over yet! My guest for the next three weeks is Steven Strogatz, a professor of applied mathematics at Cornell University and the author of "The Calculus of Friendship: What a Teacher and a Student Learned about Life While Corresponding about Math," to be published in August.
Please welcome him.
— Olivia

By Steven Strogatz

As one of Olivia Judson's biggest fans, I feel honored and a bit giddy to be filling in for her. But maybe I should confess up front that, unlike Olivia and the previous guest writers, I'm not a biologist, evolutionary or otherwise. In fact, I'm (gasp!) a mathematician.

One of the pleasures of looking at the world through mathematical eyes is that you can see certain patterns that would otherwise be hidden. This week's column is about one such pattern. It's a beautiful law of collective organization that links urban studies to zoology. It reveals Manhattan and a mouse to be variations on a single structural theme.

The mathematics of cities was launched in 1949 when George Zipf, a linguist working at Harvard, reported a striking regularity in the size distribution of cities. He noticed that if you tabulate the biggest cities in a given country and rank them according to their populations, the largest city is always about twice as big as the second largest, and three times as big as the third largest, and so on. In other words, the population of a city is, to a good approximation, inversely proportional to its rank. Why this should be true, no one knows.

Even more amazingly, Zipf's law has apparently held for at least 100 years. Given the different social conditions from country to country, the different patterns of migration a century ago and many other variables that you'd think would make a difference, the generality of Zipf's law is astonishing.

Keep in mind that this pattern emerged on its own. No city planner imposed it, and no citizens conspired to make it happen. Something is enforcing this invisible law, but we're still in the dark about what that something might be.

Many inventive theorists working in disciplines ranging from economics to physics have taken a whack at explaining Zipf's law, but no one has completely solved it. Paul Krugman, who has tackled the problem himself, wryly noted that "the usual complaint about economic theory is that our models are oversimplified — that they offer excessively neat views of complex, messy reality. [In the case of Zipf's law] the reverse is true: we have complex, messy models, yet reality is startlingly neat and simple."

After being stuck for a long time, the mathematics of cities has suddenly begun to take off again. Around 2006, scientists started discovering new mathematical laws about cities that are nearly as stunning as Zipf's. But instead of focusing on the sizes of cities themselves, the new questions have to do with how city size affects other things we care about, like the amount of infrastructure needed to keep a city going.

For instance, if one city is 10 times as populous as another one, does it need 10 times as many gas stations? No. Bigger cities have more gas stations than smaller ones (of course), but not nearly in direct proportion to their size. The number of gas stations grows only in proportion to the 0.77 power of population. The crucial thing is that 0.77 is less than 1. This implies that the bigger a city is, the fewer gas stations it has per person. Put simply, bigger cities enjoy economies of scale. In this sense, bigger is greener.

The same pattern holds for other measures of infrastructure. Whether you measure miles of roadway or length of electrical cables, you find that all of these also decrease, per person, as city size increases. And all show an exponent between 0.7 and 0.9.

Now comes the spooky part. The same law is true for living things. That is, if you mentally replace cities by organisms and city size by body weight, the mathematical pattern remains the same.

For example, suppose you measure how many calories a mouse burns per day, compared to an elephant. Both are mammals, so at the cellular level you might expect they shouldn’t be too different. And indeed, when the cells of 10 different mammalian species were grown outside their host organisms, in a laboratory tissue culture, they all displayed the same metabolic rate. It was as if they didn’t know where they’d come from; they had no genetic memory of how big their donor was.

But now consider the elephant or the mouse as an intact animal, a functioning agglomeration of billions of cells. Then, on a pound for pound basis, the cells of an elephant consume far less energy than those of a mouse. The relevant law of metabolism, called Kleiber’s law, states that the metabolic needs of a mammal grow in proportion to its body weight raised to the 0.74 power.

This 0.74 power is uncannily close to the 0.77 observed for the law governing gas stations in cities. Coincidence? Maybe, but probably not. There are theoretical grounds to expect a power close to 3/4. Geoffrey West of the Santa Fe Institute and his colleagues Jim Brown and Brian Enquist have argued that a 3/4-power law is exactly what you’d expect if natural selection has evolved a transport system for conveying energy and nutrients as efficiently and rapidly as possible to all points of a three-dimensional body, using a fractal network built from a series of branching tubes — precisely the architecture seen in the circulatory system and the airways of the lung, and not too different from the roads and cables and pipes that keep a city alive.

These numerical coincidences seem to be telling us something profound. It appears that Aristotle’s metaphor of a city as a living thing is more than merely poetic. There may be deep laws of collective organization at work here, the same laws for aggregates of people and cells.

The numerology above would seem totally fortuitous if we hadn’t viewed cities and organisms through the lens of mathematics. By abstracting away nearly all the details involved in powering a mouse or a city, math exposes their underlying unity. In that way (and with apologies to Picasso), math is the lie that makes us realize the truth.

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NOTES:

For Zipf's law see:

Zipf, G. K. (1949) "Human Behavior and the Principle of Least Effort." Addison-Wesley, Cambridge, MA.

Gabaix, X. (1999) "Zipf's law for cities: An explanation." The Quarterly Journal of Economics 114, 739–767.

For Paul Krugman quote:

Krugman, P. (1996) "Confronting the mystery of urban hierarchy." Journal of the Japanese and International Economies 10, 399–418.

The new laws of infrastructure for cities are reported in:

Bettencourt, L. M.A., Lobo, J., Helbing, D., Kühnert, C, and West, G. B. (2007) "Growth, innovation, and the pace of life in cities." Proceedings of the National Academy of Sciences 104, 7301–7306.

For an overview of Kleiber's law and the theory of West, Brown and Enquist, see:

Whitfield, J. (2006) "In the Beat of a Heart: Life, Energy, and the Unity of Nature." Joseph Henry Press, Washington DC.

For the tissue culture results about mammalian cells, see:

Brown, M. F., Gratton, T. P., and Stuart, J. A. (2007) "Metabolic rate does not scale with body mass in cultured mammalian cells." Am J Physiol Regul Integr Comp Physiol 292, R2115–R2121.

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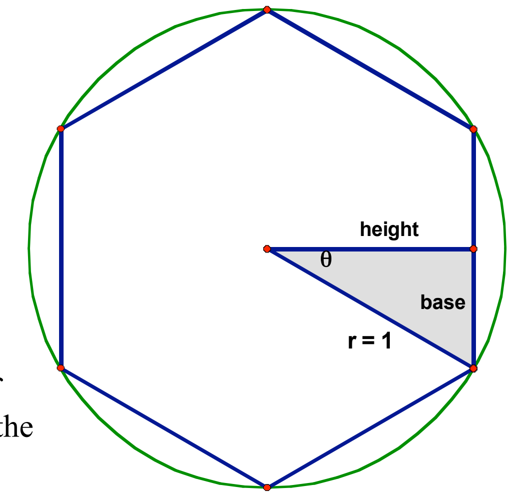
Exploration 1: Pi as a Limit

In this exploration, you will use polygons inscribed in a circle of radius $r = 1$ to estimate the value of π. As the polygon's number of sides grows, so should the accuracy of your estimate of π.

Steps



1. Find the area of the circle.

2. First consider a hexagon inscribed in the circle.
   Without a protractor, find the measure of angle θ.
   Remember, the entire circle covers 2π radians.
   (Yes, work in radians, not degrees.)

3. Use trigonometry to find the height, base, and resulting area of the shaded triangle. Don't round off any measurements; keep the sine and cosine expressions.

4. Now find the area of the entire hexagon.
   That's your estimate of π. It should be low. Why?

5. Improve your estimate by using a 12-sided polygon instead of a hexagon.
   Sketch the new shaded triangle and repeat the previous three steps.
   Preserve accuracy. Don't convert to decimals until the final step (4).
   The area of the resulting 12-sided polygon should be closer to π. Is it?

6. Repeat five more times, using polygons of successively more sides.
   Show your calculations, and present your results in a table: number of sides in column one and resulting areas in column two. Carry all decimals.

7. Generalize your work for an n-sided polygon.
   In other words, express the polygon's area as a function of n.
   Your expression will involve sine and cosine.
   Type your expression into Y= in your calculator. Of course, you'll have to use x, not n.
   Be sure your calculator is in radian mode.

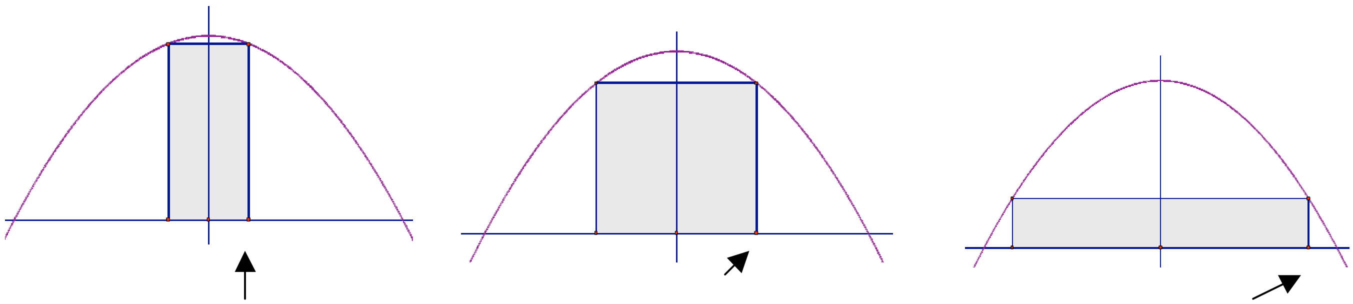
8. Look at your calculator's table, starting with $n = 3$ and incrementing by $\Delta n = 1$.
   Do your earlier calculations appear? They should!
   As n grows, what value does your expression seem to approach?
   Will it ever reach its destination? Look at large values of n to be sure.
   How accurate is your estimate of π when n equals 100 or 1000 or 10,000?

9. Look at the graph of your expression and sketch it in your write-up.
   A good viewing window is $3 \leq x \leq 20$ and $0 \leq y \leq 4$.
   Your graph should exhibit a horizontal asymptote. Explain its significance.

Exploration 2: Maximizing Area

In this exploration, you will find the rectangle of largest area that can be inscribed in the parabola $y = 9 - \frac{1}{10}x^2$. Shown below are three such rectangles.

Note that each rectangle is symmetric about the y-axis.
A small x-value (in the lower-right corner) produces a tall, thin rectangle.
A large x-value (in the lower-right corner) produces a short, wide rectangle.



$x = 2$ $\quad$ $x = 4$ $\quad$ $x = 8$

Steps

1. Find the length, height, and resulting area of each of the three rectangles shown. Don't *estimate* each height. Use the parabolic function to find each height *precisely*.

2. Repeat step 1 with several other x-values. Choose x-values that you think will result in even *larger* rectangular areas. Try to converge on the particular x-value that *maximizes* the rectangular area.

3. Present your calculations clearly in a two-column table (x-values in column one and resulting areas in column two) and plot your data on graph paper.

4. Generalize your calculations for *any* x-value. In other words, express the rectangular area as a function of x. Graph that area function on your calculator. Use a window similar to that from your hand-drawn graph.

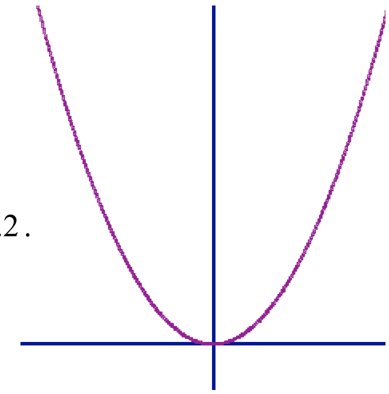
5. Trace along the area function and also look at its table. Start your table at $x = 0$ and increment by small amounts, such as $\Delta x = .1$ Do your earlier calculations appear? They should!

6. Find the x-value that maximizes the area function. Do this either by tracing along the curve, or by using Calc-Maximum while viewing the graph.

<u>Exploration 3</u>: Slope Patterns on a Curve

In this exploration, you will find the instantaneous slopes at various points along the curve $f(x) = 3x^2$ and look for a pattern to those slopes. Unlike the slopes along a line, the slopes along this curve are continually changing.

$f(x) = 3x^2$



<u>Steps</u>

Find the slope at $x = 1$ this way:

1. First, calculate the slope from $x_1 = 1$ to a second point nearby, $x_2 = 1.2$.
   Use f to get precise y-values to calculate the slope.
   Then use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to calculate the slope.

2. Improve your result by moving the second point much closer and closer to $x_1 = 1$.
   For example, let $x_2 =$ 1.02, then 1.002, then 1.0002.

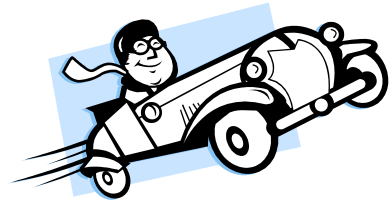
3. You now have five estimates of the instantaneous slope at $x = 1$, each better than the one before. Look at those estimates. Are they approaching a particular value?
   If so, predict the *true* instantaneous slope at $x = 1$.

4. Repeat steps 1-3 to find the curve's instantaneous slope at $x = 2$, 3, and 4.
   The instantaneous slopes will grow, since the curve gets steeper as x moves east.

5. Now look at your instantaneous slopes for $x = 1$, 2, 3, and 4. Do you see a pattern? You should! Use the pattern to predict the instantaneous slope at $x = 10$ and at $x = -7$.
   Use the parabola's graph to explain why your predictions make sense.

6. Generalize your findings: find an expression for the slope at *any* x-value.
   That expression is called the *derivative* of $f(x)$ and is denoted as $f'(x)$.
   It's pronounced "f prime of x."

7. Now consider again the area function from exploration 2: $A(x) = 18x - \frac{1}{5}x^3$. Its derivative happens to be: $A'(x) = 18 - \frac{3}{5}x^2$. A function's *maximum value* occurs where its *derivative equals zero*. So, solve algebraically the equation $A'(x) = 0$ to find when $A(x)$ is maximized. Remember, after finding x, use it to evaluate $A(x)$.
   Do your results confirm your work in exploration 2? They should!



<u>Driving Project</u>

This project involves taking a 30-minute car ride, collecting data, and analyzing results.
You may work alone or with another calculus student.
If you work with another student, the two of you may submit just one write-up.

Collecting Data

- Record your car's odometer reading *before* starting your car ride.
- With another person* driving, take a 30-minute car ride in a residential area. Find a route that allows you to change speeds throughout the trip.
- During the trip, record your velocity every two minutes ($\frac{1}{30}th$ of an hour). When done, you should have 16 pieces of data for $t = \frac{0}{30}, \frac{1}{30}, \frac{2}{30}, \cdots, \frac{15}{30}$ hours.
- Record the odometer reading *after* your trip.

Analyzing Results

1. Present your data in a table and plot velocity vs. time (in hours) on graph paper. Connect your data points with line segments to form your velocity graph. Clearly identify what each axis represents and include its measurement units. Also identify the increments on each axis.

2. Working from left to right, calculate the area under your velocity graph for each two-minute time interval. You will calculate 15 areas. Most will be trapezoids, but some might be rectangles or triangles. Do you remember the area of a trapezoid?

$$Area = \tfrac{1}{2}(b_1 + b_2)h \qquad \text{(The bases are the parallel sides.)}$$

3. Add your fifteen areas to find the *total* area under your velocity graph.

4. In a few well-written paragraphs, answer these questions:

   - How does the total area under your graph compare to the two odometer readings?

   - What does the area under your velocity graph *represent*? Address units. Divide the total area by ½ hour. What does *that* result represent? Include units.

   - What does the slope during each 2-minute time interval represent? Include units. What does a negative slope imply about your motion?

   - Is your velocity graph a *complete* picture of how your velocity changed during your trip? If not, how could you make it more complete? What other insights can you draw from your results and analysis?

*Another person - a parent would be a good idea!

Topic 11: Complex fractions

Simplify the following

1. $\dfrac{x}{x-\frac{1}{2}}$

2. $\dfrac{\frac{1}{x}+4}{\frac{1}{x}-2}$

3. $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$

4. $\dfrac{\frac{3}{x}-\frac{4}{y}}{\frac{4}{x}-\frac{3}{y}}$

5. $\dfrac{1-\frac{2}{3x}}{x-\frac{4}{9x}}$

6. $\dfrac{\frac{x^2-y^2}{xy}}{\frac{x+y}{y}}$

7. $\dfrac{x^{-3}-x}{x^{-2}-1}$

8. $\dfrac{\frac{x}{1-x}+\frac{1+x}{x}}{\frac{1-x}{x}+\frac{x}{1+x}}$

9. $\dfrac{\frac{4}{x-5}+\frac{2}{x+2}}{\frac{2x}{x^2-3x-10}+3}$

Topic 12: Composition of functions

If $f(x) = x^2$, $g(x) = 2x - 1$, and $h(x) = 2^x$, find the following

1. $f(g(2))$

2.. $f(g(2))$

3. $f(h(-1))$

4. $h(f(-1))$

5. $g\left(f\left(h\left(\frac{1}{2}\right)\right)\right)$

6. $f(g(x))$

7. $g(f(x))$

8. $g(g(x))$

9. $f(h(x))$

Topic 13: Solving Rational (fractional) equations

Solve each equation for x

1. $\frac{2}{3} - \frac{5}{6} = \frac{1}{x}$

2. $x + \frac{6}{x} = 5$

3. $\frac{x+1}{3} - \frac{x-1}{2} = 1$

4. $\frac{x-5}{x+1} = \frac{3}{5}$

5. $\frac{60}{x} - \frac{60}{x-5} = \frac{2}{x}$

6. $\frac{2}{x+5} + \frac{1}{x-5} = \frac{16}{x^2-25}$

7. $\frac{x}{x-2} + \frac{2x}{4-x^2} = \frac{5}{x+2}$

8. $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

9. $\frac{2x+3}{x-1} = \frac{10}{x^2-1} + \frac{2x-3}{x+1}$

Topic 15: Solving Trigonometric equations

Solve each equation on the interval $\left[0, 2\pi\right)$

1. $\sin x = \frac{1}{2}$

2. $\cos^2 x = \cos x$

3. $2\cos x + \sqrt{3} = 0$

4. $4\sin^2 x = 1$

5. $2\sin^2 x + \sin x = 1$

6. $\cos^2 x + 2\cos x = 3$

7. $2\sin x \cos x + \sin x = 0$

8. $8\cos^2 x - 2\cos x = 1$

9. $\sin^2 x - \cos^2 x = 0$