B.4 Solving Inequalities Algebraically and Graphically

Properties of Inequalities

The inequality symbols $<$, $\leq$, $>$, and $\geq$ are used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. These values are **solutions** of the inequality and are said to **satisfy** the inequality. For example, the number 9 is a solution to

$$5x - 7 > 3x + 9$$

because when you substitute $x = 9$,

$$5(9) - 7 > 3(9) + 9 \qquad \textbf{Substitute } \boldsymbol{x = 9}$$

$$45 - 7 > 27 + 9$$

$38 > 36$ is a true statement.

Properties of Inequalities

The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality.

The set of all **points** on the real number line that represent the solution set is the **graph of the inequality**. Graphs of many types of inequalities consist of intervals on the real number line.

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable you can make use of the **properties of inequalities**. These properties are similar to the properties of equality, but there are two important exceptions.

1. When each side of an inequalities is multiplied or divided by a negative number, *the direction of the inequality symbol must be reversed* in order to maintain a true statement.
2. Two inequalities that have the same solution set are **equivalent inequalities**.

Properties of Inequalities

1. When each side of an inequalities is <u>multiplied or divided by a negative number</u>, *the direction of the inequality <u>symbol must be reversed</u>* in order to maintain a true statement.

$$-2 < 5$$

$$(-3)(-2) > (-3)(5) \quad \textbf{Reverse sign, Multiply by -3}$$

$$6 > -15$$

2. Two inequalities that have the same solution set are **equivalent inequalities**.

$$x + 2 < 5 \quad \text{and} \quad x < 3$$

$$x + 2 - 2 < 5 - 2 \quad \textbf{Subtract 2 from both sides}$$

$$x < 3$$

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. Transitive Property

   if $a < b$ and $b < c$ then $a < c$

2. Addition of Inequalities

   if $a < b$ and $c < d$ then $a + c < b + d$

3. Addition of a Constant

   if $a < b$ then $a + c < b + c$

4. Multiplying by a Constant

for $c > 0$, if $a < b$ then $ac < bc$

for $c < 0$, if $a < b$ then $ac > bc$

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Solving a Linear Equality

Example 1

Solve

$5x - 7 > 3x + 9$

Solving a Linear Equality

Algebraic Solution:

$5x - 7 > 3x + 9$

$-3x \qquad -3x$ **Subtract -3x from both sides**

$2x - 7 > 9$

$+7 \quad +7$ **Add 7 to both sides**

$\frac{2x}{2} > \frac{16}{2}$ **Divide both sides by 2**

$x > 8$

So, the solution set is all real numbers that are greater than 8. The **interval notation** for this solution set is (8, ∞)

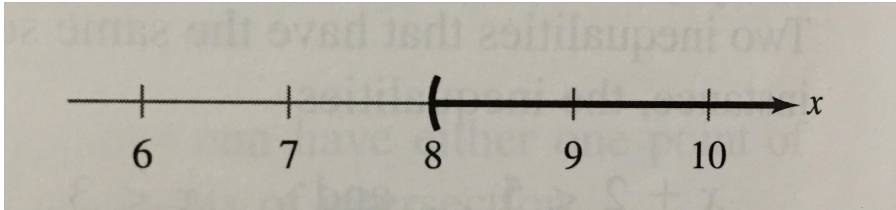


Figure B.55 Solution interval: $(8, \infty)$

Solving an Inequality

Example 2

Solve

$1 - (3/2)x \geq x - 4$

Solving an Inequality

Algebraic Solution:

$$1 - (3/2)x \geq x - 4$$

$$2[1 - (3/2)x] \geq 2[x - 4]$$ **Multiply each side by the LCD**

$$2 - 3x \geq 2x - 8$$

$$10 - 3x \geq 2x$$ **Add 8 to both sides**

$$10 \geq 5x$$ **Divide both sides by 2**

$$2 \geq x$$

The solution set is all real numbers that are less than or equal to 2. The interval notation for this solution set is [- ∞, 2].

- What would this look like on a number line?
- Try plugging in a number less 2 into the original equation.

Solving an Inequality

Graphical Solution

$1 - (3/2)x \geq x - 4$

Let $y_1 = 1 - (3/2)x$ and $y_2 = x - 4$

You can see that the *point of intersection* is (2, -2).

The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, which implies $y_1 \geq y_2$ for all $x \leq 2$

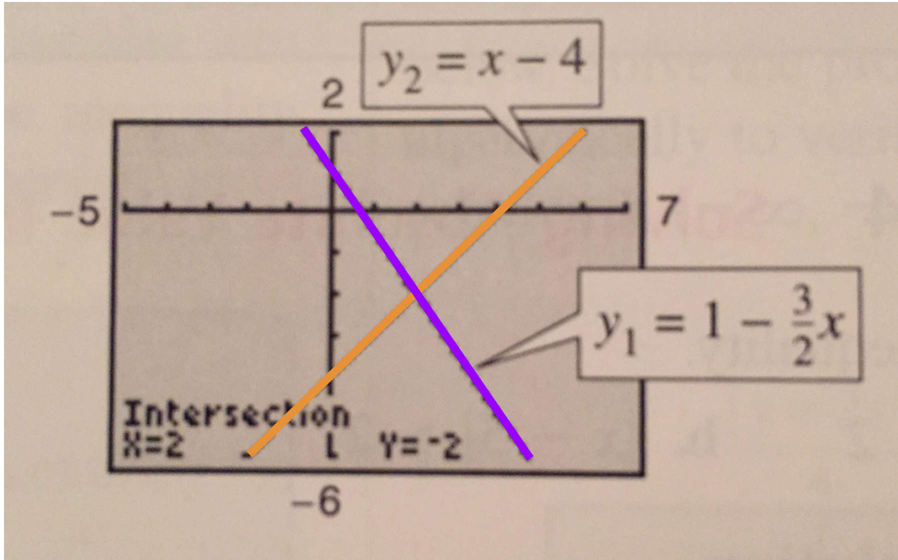


Figure B.57

Solving a Double Inequality

Example 3

Solve

$-3 \leq 6x - 1$ and $6x - 1 < 3$

What would the interval notation be?

Solving a Double Inequality

Algebraic Solution

$-3 \leq 6x - 1$ and $6x - 1 < 3$

$-3 \leq 6x - 1 < 3$ **Write as a double Inequality**

$-2 \leq 6x < 4$ **Add 1 to each part**

$-\frac{1}{3} \leq x < \frac{2}{3}$ **Divide by 6 and simplify**

The interval notation for this solution set is $[-\frac{1}{3}, \frac{2}{3})$

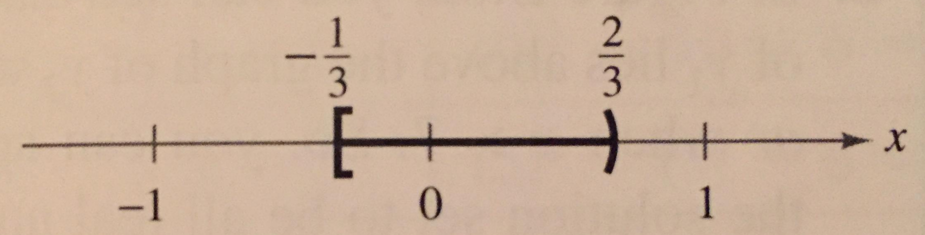


Figure B.58 Solution interval $\left[-\frac{1}{3}, \frac{2}{3}\right)$

Solving a Double Inequality

Graphical Solution

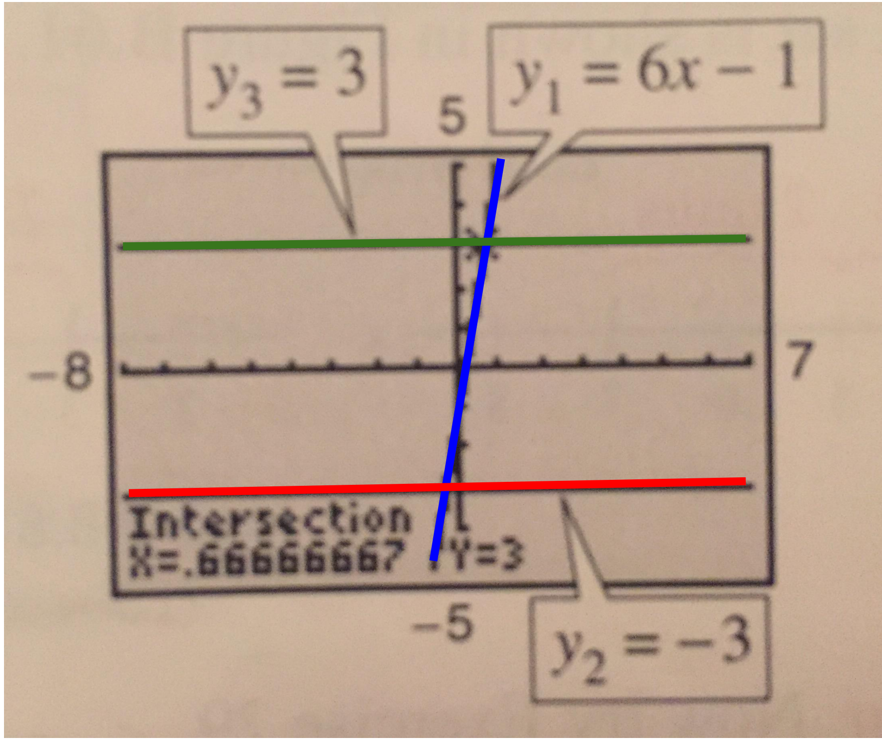
Let $y_1 = 6x - 1$ **$y_2 = -3$**

$y_3 = 3$

Use the *intersect* feature to find that the points of intersection are ($-\frac{1}{3}$, -3) and ($\frac{2}{3}$, 3).

The graph of **y_1** lies above the graph of **y_2** to the right of ($-\frac{1}{3}$, -3) ***AND*** the graph of **y_1** lies below the graph of **y_3** to the left of ($\frac{2}{3}$, 3).

This implies that **$y_2 \leq y_1 < y_3$** when $-\frac{1}{3} \leq x < \frac{2}{3}$



Inequalities Involving Absolute Value

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between -a and a

   $|x| < a$ if and only if $-a < x < a$ **Double inequality**

2. The solutions of $|x| > a$ are all values of x that are less than -a or greater than a.

   $|x| > a$ if and only if $x < -a$ or $x > a$ **Compound inequality**

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

- What would each of these look like on a number line?

Solving a Double Inequality

Example 4

Solve

a. $|x - 5| < 2$

b. $|x - 5| > 2$

What would the interval notation be?

Use graphing calculator to graph each inequality.

Solving a Double Inequality

Algebraic Solution

a. $|x - 5| < 2$

$-2 < x - 5 < 2$ **Write the double inequality**

$3 < x < 7$ **Add 5 to each part**

The interval notation for this solution set is (3, 7).

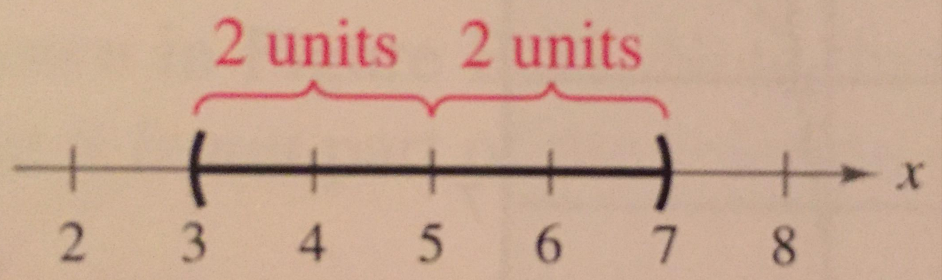


Figure B.60

Solving a Double Inequality

Graphical Solution

a. $|x - 5| < 2$

Let $y_1 = |x - 5|$ and $y_2 = 2$

Use the *intersect* feature on your graphing calculator.

The points of intersection are (3, 2) and (7, 2).

The graph of y_1 lies below the graph of y_2 when $3 < x < 7$.

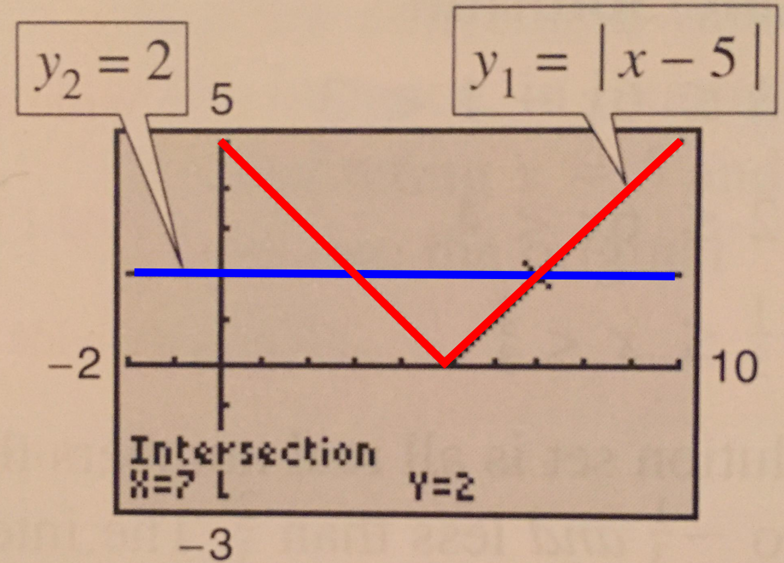


Figure B.62

Solving a Double Inequality

Algebraic Solution

b. $|x - 5| > 2$

$x - 5 < -2$ or $x - 5 > 2$

Solve first inequality: $x - 5 < -2$

$x < 3$ **Add 5 to each side**

Solve second inequality: $x - 5 > 2$

$x > 7$ **Add 7 to each side**

The interval notation for this solution set is $(-\infty, 3) \cup (7, \infty)$

The symbol $\cup$ is called a ***union*** symbol and is used to denote the combining of two sets.

Solving a Double Inequality

Graphical Solution

b. $|x - 5| > 2$

Let $y_1 = |x - 5|$ and $y_2 = 2$

The points of intersection are (3, 2) and (7, 2).

The graph of y_1 lies above the graph of y_2 when $x < 3$ or when $x > 7$

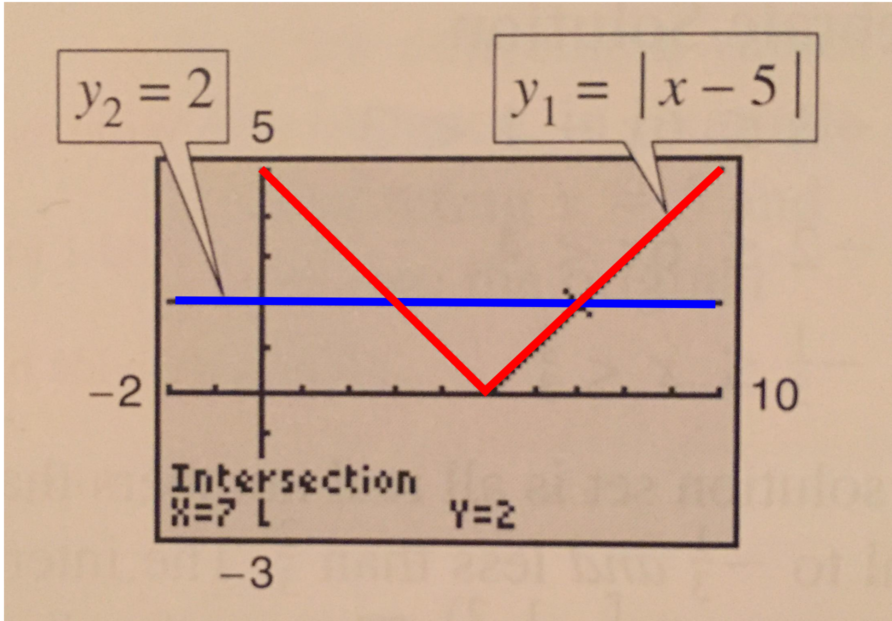


Figure B.62

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 > 0$, use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero).

These zeros are the **critical numbers** of the inequality, and the resulting open interval are the **test intervals** for the inequality. For example,

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$, which divide the real number line into three test intervals: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$.

To solve the inequality $x^2 - 2x - 3 > 0$, you need to test **only one value** from **each test interval**.

Polynomial Inequalities

<u>Finding Test Intervals for a Polynomial</u>

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. The zeros of a polynomial are its critical numbers.

2. Use the critical numbers to determine the test intervals.

3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for *every* x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for *every* x-value in the interval.

Investigating Polynomial Behavior

To determine the intervals on which x^2 - x - 6 is entirely negative and those on which it is entirely positive. factor the quadratic as x^2 - x - 6 = (x +2)(x - 3).

The critical numbers occur at x = -2 and x = 3.

The test intervals for the quadratic are (-∞, -2), (-2, 3), and (3, ∞).

In each test interval, choose a representative x-value and evaluate the polynomial.

Interval	x-Value	Value of Polynomial	Sign of Polynomial
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 5$	$(5)^2 - (5) - 6 = 14$	Positive

The polynomial had negative values for every x in the interval (-2, 3) and positive values for every x in the intervals (-∞, -2) and (3, ∞).

Investigating Polynomial Behavior

From the graph, you can see whether the graph is positive or negative on each interval.

$x^2 - x - 6 > 0$ on the intervals $(-\infty, -2)$ and $(3, \infty)$.

$x^2 - x - 6 < 0$ on the interval $(-2, 3)$.

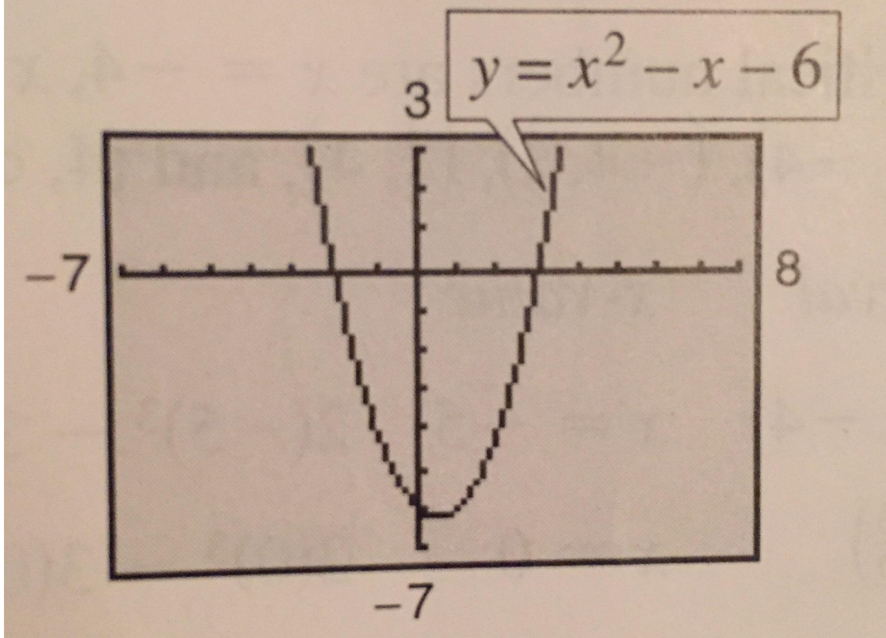


Figure B.63

Solving a Polynomial Inequality

Example 6

Solve

$2x^2 + 5x > 12$

Critical numbers?

Test interval?

Test?

Solution set?

Solving a Polynomial Inequality

Algebraic Solution

$2x^2 + 5x > 12$

$2x^2 + 5x - 12 > 0$

$(x + 4)(2x - 3) > 0$

***Critical Numbers*:** $x = -4, 3/2$

Test Intervals: $(-\infty, -4)$, $(-4, 3/2)$, and $(3/2, \infty)$

Test: Is $(x + 4)(2x - 3) > 0$?

After testing these intervals, you can see that the polynomial $2x^2 + 5x - 12$ is positive on the open intervals $(-\infty, -4)$ and $(3/2, \infty)$.

Therefore the solution set of the inequality is

$(-\infty, -4) \cup (3/2, \infty)$

Solving a Polynomial Inequality

Graphical Solution

$2x^2 + 5x > 12$

$2x^2 + 5x - 12 > 0$

Graph $y = x^2 + 5x - 12$.

You can see that the graph is *above* the x-axis when x is less than -4 or when x is greater than 3/2. So you can graphically approximate the solution set to be

$(-\infty, -4) \cup (3/2, \infty)$

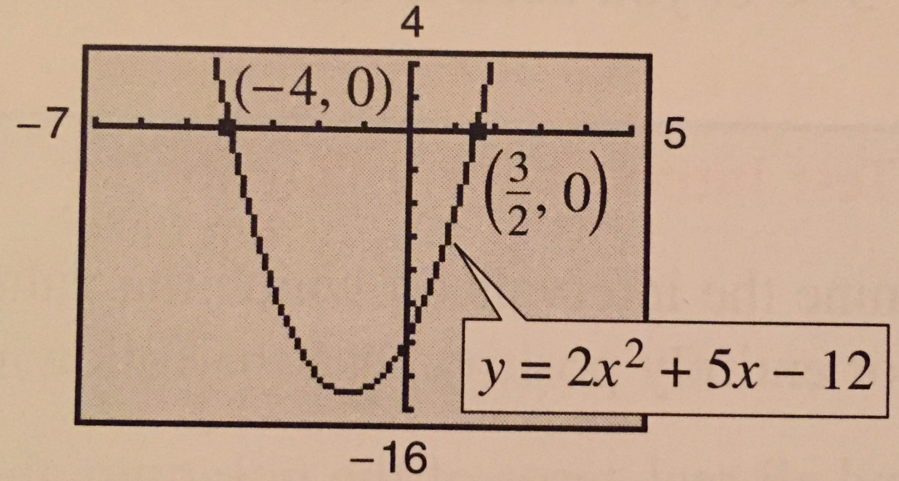


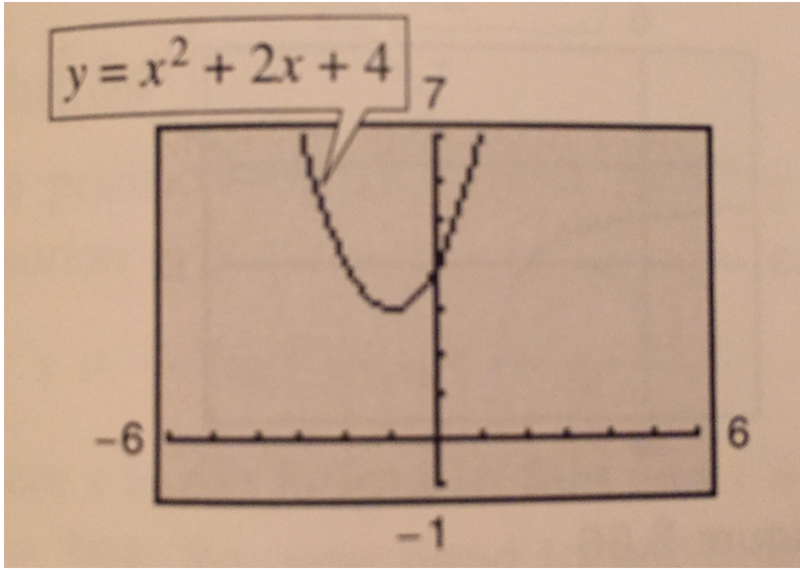
Figure B.64

Unusual Solution Sets

a. The solution set of

$x^2 + 2x + 4 > 0$

consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x.



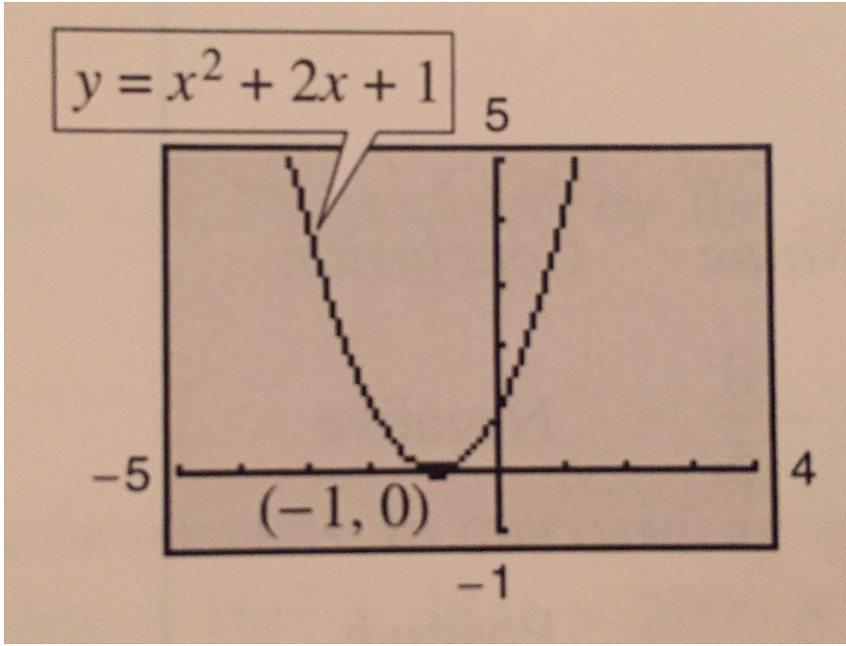
Unusual Solution Sets

b. The solution set of

$x^2 + 2x + 1 \le 0$

$(x + 1)^2 \le 0$

consists of the single real number {-1}, because the quadratic $x^2 + 2x + 1$ has one critical number, $x = -1$, and it is the only value that satisfies the inequality.

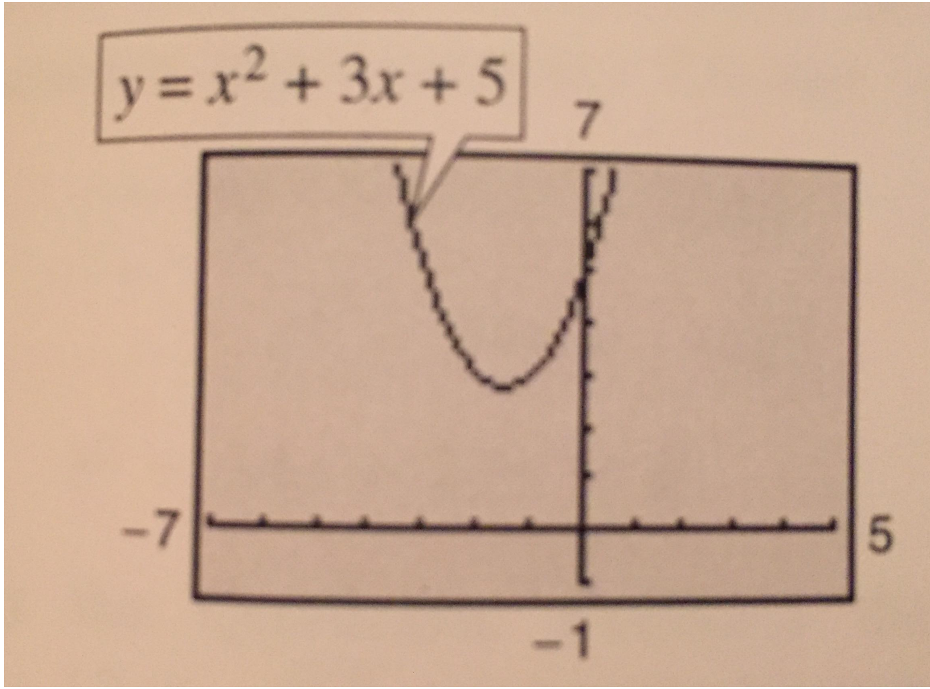


Unusual Solution Sets

c. The solution set of

$x^2 + 3x + 5 < 0$

is empty. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x.



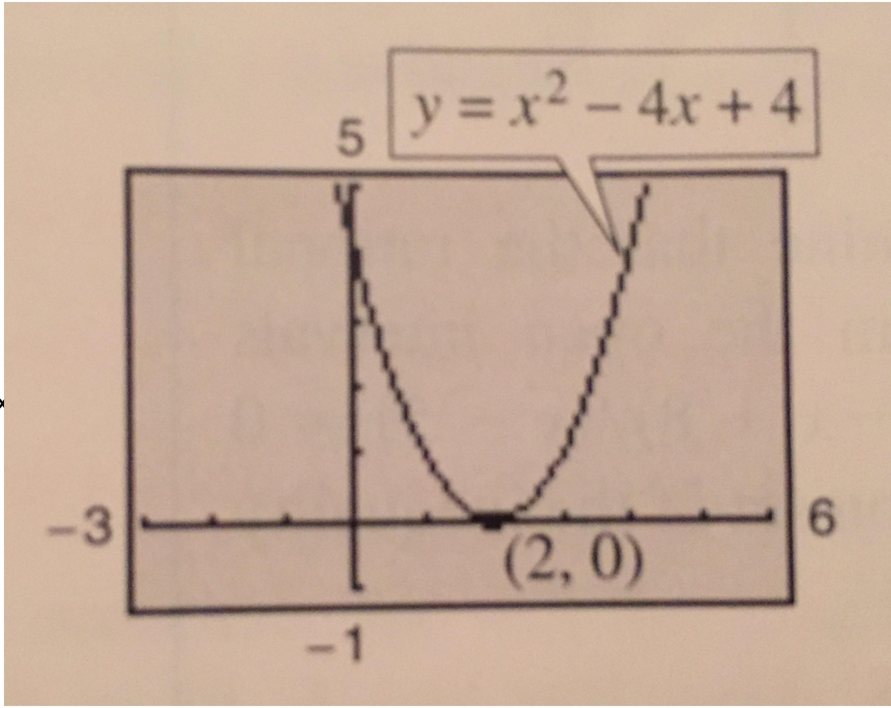
Unusual Solution Sets

d. The solution set of

$x^2 - 4x + 4 > 0$

$(x - 2)^2 > 0$

consists of all real numbers *except* the number 2. In interval notation, this solution set can be written as $(-\infty$ $-2) \cup (2, \infty)$. The graph of $x^2 - 4x + 4$ lies above the x-axis except as x = 2, where it touches it.



Rational Inequalities

The concepts of **critical numbers** and **test intervals** can be extended to inequalities involving rational expressions.

To do this, use the fact that the value of a rational expression can change sign only at its **zeros** (the x-values for which its **numerator is zero**) and its ***undefined values*** (the x-values for which its **denominator is zero**).

These two types of numbers make up the ***critical numbers*** of a rational inequality.

Solving a Polynomial Inequality

Example 9

Solve

$$\frac{2x - 7}{x - 5} \leq 3$$

Critical numbers?

Test interval?

Test?

Solution set?

Solving a Polynomial Inequality

Algebraic Solution

$$\frac{2x - 7}{x - 5} \leq 3$$

$$\frac{2x - 7}{x - 5} - 3 \leq 0$$ **Write in general form**

$$\frac{2x - 7 - 3x + 15}{x - 5} \leq 0$$ **Write as single fraction**

$$\frac{-x + 8}{x - 5} \leq 0$$

You can see that the critical numbers are x = 5 and x = 8

Solving a Polynomial Inequality

Critical Numbers: x = 5, x = 8

Test Intervals: (-∞, 5), (5, 8), (8, ∞)

Test: Is $\dfrac{-x + 8}{x - 5} \leq 0$?

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
(-∞, 5)	x = 0	$\dfrac{-(0) + 8}{(0) - 5} = \dfrac{-8}{5}$	Negative
(5, 8)	x = 6	$\dfrac{-(6) + 8}{(6) - 5} = \dfrac{2}{1}$	Positive
(8, ∞)	x = 9	$\dfrac{-(9) + 8}{(9) - 5} = \dfrac{-1}{4}$	Negative

Because (-x + 8)/(x - 5) = 0 when x = 8, the solution set of the inequality is (-∞, 5) ∪ [8, ∞).

Solving a Polynomial Inequality

Graphical Solution

$y_1 = \dfrac{2x - 7}{x - 5}$ $\qquad$ $y_2 = 3$

Use the intersect feature to confirm the intersection point of (8, 3). The graph of y_1 lies below the graph of y_2 in the interval $(-\infty, 5)$ and $[8, \infty)$.

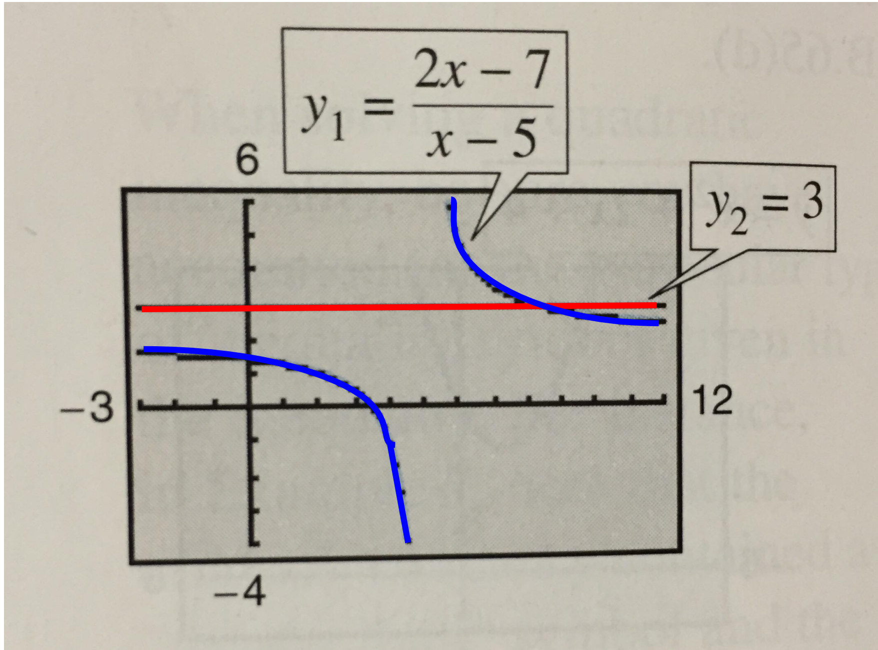


Figure B.66

Finding the Domain of an Expression

Example 10

Find the domain of $\sqrt{(64 - x^2)}$

Critical numbers?

Test Interval?

Test?

Solution set?

Finding the Domain of an Expression

Solution

Find the domain of $\sqrt{(64 - x^2)}$

Because $\sqrt{(64 - x^2)}$ is defined only if $64 - 4x^2$ is **nonnegative**, the domain is given by

$$64 - 4x^2 \geq 0$$

$$16 - x^2 \geq 0 \qquad \textbf{Divide each side by 4}$$

$$(4 - x)(4 + x) \geq 0 \qquad \textbf{Factor}$$

The inequality has two critical numbers: $x = -4$ and $x = 4$. A test shows that $64 - 4x^2 \geq 0$ in the ***closed interval*** $[-4, 4]$. The graph of $y = \sqrt{(64 - x^2)}$ confirms that the domain is $[-4, 4]$.

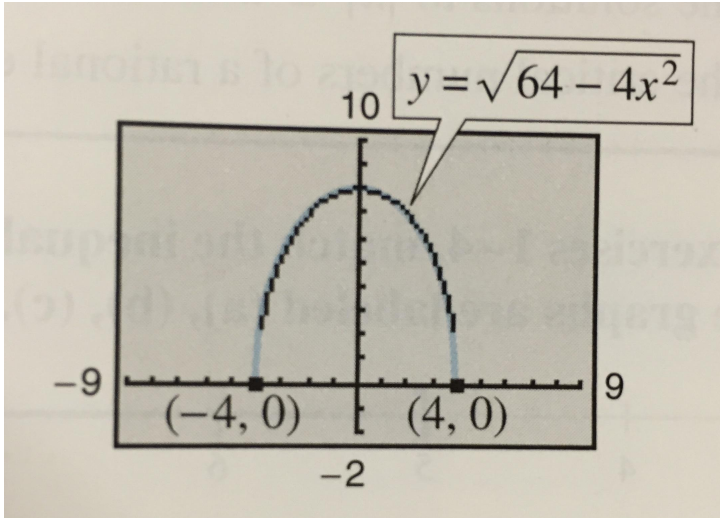


Figure B.67

Height of a Projectile

Example 11

A projectile is fired straight upward from ground level with an initial velocity of 384 feet per second. During what time period will its height exceed 2000 feet?

The position of an object moving vertically can be modeled by the *position equation*.

$$s = -16t^2 + v_0t + s_0$$

where s is the height in feet and t is the time in seconds. v_0 represents the velocity at t = 0 and s_0 represents the height at t = 0.

Height of a Projectile

Solution

$$s = -16t^2 + v_0t + s_0$$

Since the projectile is fired from ground level, $s_0 = 0$ because the height at $t = 0$ is 0.

Since the projectile is fired with an initial velocity of 384 feet per second, $v_0 = 384$.

We want to find when the position of the projectile is above 2000 feet so we need to solve the inequality:

$$-16t^2 + 384t > 2000$$

Height of a Projectile

Solution

$s = -16t^2 + v_0t + s_0$

$-16t^2 + 384t > 2000$

Let $y_1 = -16t^2 + 384t$ **$y_2 = 2000$**

Use the intersect feature to approximate the x-values of the intersections points to be x = 7.6 and x = 16.4 and you can see it is between these values that $-16t^2 + 384t > 2000$. So your solution set is (7.6, 16.36).

What does this mean in terms of what was originally asked?

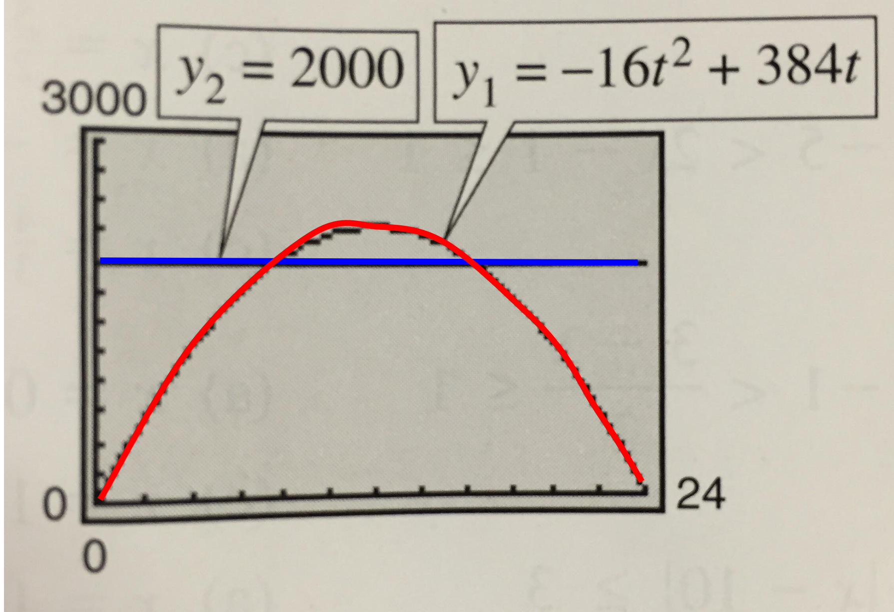


Figure B.68

Homework

- B.4 Homework due next class
- Unit 1 Test Review next class
- Unit 1 Test the following class.